A Linear-Time Algorithm for the Maximum Matched-Paired-Domination Problem in Cographs

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Abstract

Let $G = (V, E)$ be a graph without isolated vertices. A matching in G is a set of independent edges in G. A perfect matching M in G is a matching such that every vertex of G is incident to an edge of M. A set $S \subseteq V$ is a *paired-dominating set* of G if every vertex in $V - S$ is adjacent to some vertex in S and if the subgraph $G[S]$ induced by S contains at least one perfect matching. The paired-domination problem is to find a paired-dominating set of G with minimum cardinality. In this paper, we introduce a generalization of the paired-domination problem, namely the maximum matched-paired-domination problem. A set $MPD \subseteq E$ is a *matched-paired-dominating set* of G if MPD is a perfect matching of $G[S]$ induced by a paired-dominating set S of G. Note that the paired-domination problem can be regard as finding a matched-paired-dominating set of G with minimum cardinality. Let $\mathcal{R}$ be a subset of V, MPD be a matched-paired-dominating set of G, and let $V(MPD)$ denote the set of vertices being incident to edges of MPD. A *maximum matched-paired-dominating set* $MMPD$ of G w.r.t. $\mathcal{R}$ is a matched-paired-dominating set such that $|V(MMPD) \cap \mathcal{R}| \geqslant |V(MPD) \cap \mathcal{R}|$. An edge in MPD is called *free-paired-edge* if neither of its both vertices is in $\mathcal{R}$. Given a graph G and a subset $\mathcal{R}$ of vertices of G, the *maximum matched-paired-domination problem* is to find a maximum matched-paired-dominating set of G with the least free-paired-edges; note that, if $\mathcal{R}$ is empty, the stated problem coincides with the classical paired-domination problem. In this paper, we present a linear-time algorithm to solve the maximum matched-paired-domination problem in cographs.

Keywords. graph algorithm, linear-time algorithm, paired-domination, maximum matched-paired-domination, cographs

AMS subject classifications. 05C69, 05C85, 68Q25

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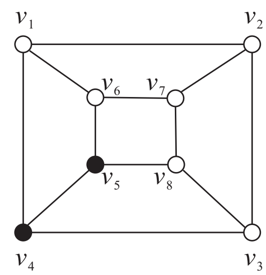


Fig. 1: The tree-cube graph Q_3, where restricted vertices are drawn by filled circles.

1 Introduction

All graphs considered in this paper are finite and undirected, without loops or multiple edges. Let $G = (V, E)$ be a graph without isolated vertices. The *open neighborhood* $N_G(v)$ of the vertex v in G is defined to be $N_G(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* $N_G[v]$ of v is $N_G(v) \cup \{v\}$. For a set $S \subseteq V$, the subgraph of G induced by the vertices in S is denoted by $G[S]$. A set $D \subseteq V$ is a *dominating set* of G if every vertex not in D is adjacent to at least a vertex in D. The *domination problem* is to find a dominating set of G with minimum cardinality. The bibliography in domination and its variations maintained by Haynes et al. [13] currently has over 1200 entries; Hedetniemi and Laskar [16] edited a special issue of *Discrete Mathematics* devoted entirely to domination, and two books on domination and its variations in graphs [13, 14] have been written.

A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in a graph G is a matching such that every vertex of G is incident to an edge of M. A *paired-dominating set* of a graph G is a set PD of vertices of G such that PD is a dominating set of G and $G[PD]$ contains at least one perfect matching. In other words, a paired-dominating set with matching M is a dominating set $PD = \{v_1, v_2, \cdots, v_{2t-1}, v_{2t}\}$ with independent edge set $M = \{e_1, e_2, \cdots, e_t\}$, where each edge e_i joins two vertices of PD. The minimum cardinality of a paired-dominating set for a graph G is called the *paired-domination number*, denoted by $\gamma_{\rm p}(G)$. A paired-dominating set of G with cardinality $\gamma_{\rm p}(G)$ is called a *minimum paired-dominating set* of G. The *paired domination problem* is to find a minimum paired-dominating set of G. Note that every graph without isolated vertices contains a minimum paired-dominating set [15]. For example, for the three-cube graph Q_3 in Fig. 1, $PD = \{v_1, v_2, v_3, v_4\}$ with matching $M_1 = \{v_1v_2, v_3v_4\}$ or PD with matching $M_2 = \{v_1v_4, v_2v_3\}$ is a minimum paired-dominating set of Q_3 and $\gamma_{\rm p}(Q_3) = 4$.

Paired-domination was introduced by Haynes and Slater and the decision problem to de-

termine $\gamma_{\mathrm{p}}(G)$ of an arbitrary graph G has been known to be NP-complete [15]. It is still NP-complete on some special classes of graphs, including bipartite graphs, chordal graphs, and split graphs [6]. However, it admits polynomial time algorithms when the input is restricted to be in some special classes of graphs, including trees [23], circular-arc graphs [7], permutation graphs [8], block graphs, and interval graphs [6].

Paired-domination has found the following application [15]. In a graph G if we think of each vertex s as the possible location for a guard capable of protecting each vertex in $N_G[s]$, then "domination" requires every vertex to be protected. In paired-domination, each guard is assigned another adjacent one, and they are designed as backups for each other. However, some locations may play more important role (for example, important facilities are placed on these locations) and, hence, they are placed by guards for instant monitoring and protection possible. In this application, the number of guards placed on the important locations is as large as possible. Motivated by the above issue we introduce a generalization of the paired-domination problem, namely, the maximum matched-paired-domination problem.

Let $G = (V, E)$ be a graph without isolated vertices, $\mathcal{R}$ be a subset of V, and let PD be a paired-dominating set of G. For a set M of independent edges in G, we use $V(M)$ to denote the set of vertices being incident to edges of M. A set $MPD \subseteq E$ is called a *matched-paired-dominating set* of G if MPD is a perfect matching of $G[PD]$ induced by a paired-dominating set PD of G. That is, $V(MPD)$ is a paired-dominating set PD of G and MPD specifies a perfect matching of $G[PD]$. Note that the paired-domination problem can be regard as finding a matched-paired-dominating set of G with minimum cardinality. For an edge $e = uv \in MPD$, we say that e is a *paired-edge* in MPD, u is paired with v, and u is the *partner* of v. In addition, we will use $\langle u, v\rangle$ to denote a paired-edge uv in MPD if it is understood without ambiguity. Note that in a paired-dominating set PD of G, it is necessary to specify which vertex is the partner of a vertex in PD. The *matched number* of a matched-paired-dominating set MPD is defined to be $|V(MPD) \cap \mathcal{R}|$. The *maximum matched number* $\beta(G)$ of G is defined to be the largest matched number of a matched-paired-dominating set in G. A *maximum matched-paired-dominating set* of G w.r.t. $\mathcal{R}$ is a matched-paired-dominating set with matched number $\beta(G)$. A paired-edge in MPD is called *free-paired-edge* if both of its vertices are not in $\mathcal{R}$. A matched-paired-dominating set of G is called *canonical* if it is a maximum matched-paired-dominating set of G with the least free-paired-edges. Given a graph G and a subset $\mathcal{R}$ of vertices of G, the *maximum matched-paired-domination problem* is to find a canonical matched-paired-dominating set of G w.r.t. $\mathcal{R}$. Note that if $\mathcal{R}$ is empty, the stated problem coincides with the classical paired-domination problem. We call $\mathcal{R}$ the *restricted vertex set* of

G. The vertices in $\mathcal{R}$ are called *restricted vertices* and the other vertices are called *free vertices.* For example, given a graph G and a restricted vertex set $\mathcal{R} = \{v_4, v_5\}$ shown in Fig. 1, let $MPD_1 = \{\langle v_1, v_2\rangle, \langle v_3, v_4\rangle\}$, $MPD_2 = \{\langle v_4, v_5\rangle, \langle v_2, v_7\rangle\}$, and let $MPD_3 = \{\langle v_1, v_4\rangle, \langle v_5, v_6\rangle\}$. We can see that $\beta(G) = 2 \leqslant |\mathcal{R}|$. Then, both MPD_2 and MPD_3 are maximum matched-paired-dominating sets of G, but MPD_1 is not a maximum matched-paired-dominating set of G. It is straightforward to see that MPD_2 contains a free-paired-edge and MPD_3 contains no free-paired-edge. Thus, MPD_3 is a canonical matched-paired-dominating set of G, but MPD_2 is not canonical.

Now, we review cographs. Cographs (also called complement-reducible graphs) are defined as the class of graphs formed from a single vertex under the closure of the operations of *union* and *complement*. Cographs were introduced by Lerchs [20], who studied their structural and algorithmic properties and enumerated the class. Names synonymous with cographs include D^*-graphs, P_4 restricted graphs, and Hereditary Dacey graphs. Several characterizations of cographs are known. For example, it is shown that G is a cograph if and only if G contains no P_4 (a path consisting of four vertices) as an induced subgraph [9]. Cographs have arisen in many disparate areas of mathematics and have been independently rediscovered by various researchers. These graphs can be recognized in linear time [10, 12]. The class of cographs forms a subclass of distance-hereditary graphs [9, 10] and permutation graphs, and is a superclass of threshold graphs and complete-bipartite graphs. Numerous properties and optimization problems in these graphs have been studied [2, 3, 5, 11, 17, 18, 19, 21, 22, 24, 25, 26]. In this paper, we will solve the maximum matched-paired-domination problem on cographs in linear time.

2 Known Results and Terminology

Let G be a graph without isolated vertices. Haynes and Slater showed that a paired-dominating set of G does exist and $\gamma_{\mathrm{p}}(G)$ is even [15].

Lemma 1. *[15] Let G be a graph without isolated vertices. Then, there exists a paired-dominating set in G and $\gamma_{\mathrm{p}}(G)$ is even.*

It follows from Lemma 1 that we have the following corollary.

Corollary 2. *Let G be a graph without isolated vertices. Then, there exists a canonical matched-paired-dominating set in G.*

The following lemma is easily verified from the definition.

Lemma 3. *Assume G is a graph without isolated vertices and $\mathcal{R}$ is a restricted vertex set of G. Let MPD be a matched-paired-dominating set of G w.r.t. $\mathcal{R}$. Then,*
(1) if $|V(MPD) \cap \mathcal{R}| = |\mathcal{R}|$ and $|MPD| = \lceil \frac{|\mathcal{R}|}{2} \rceil$, then $\beta(G) = |\mathcal{R}|$ and MPD is a canonical matched-paired-dominating set of G;
(2) if $|V| = |\mathcal{R}|$ is odd, $|V(MPD) \cap \mathcal{R}| = |\mathcal{R}| - 1$, and $|MPD| = \lfloor \frac{|\mathcal{R}|}{2} \rfloor$, then $\beta(G) = |\mathcal{R}| - 1$ and MPD is a canonical matched-paired-dominating set of G.

Now, we define some notations to be used in the paper. In the following, we use $\mathcal{R}$ to denote the restricted vertex set of a graph G.

Definition 1. A paired-edge in a matched-paired-dominating set of G w.r.t. $\mathcal{R}$ is called *full-paired-edge* if both of its vertices are in $\mathcal{R}$, is called *semi-paired-edge* if its one vertex is in $\mathcal{R}$ but the other vertex is not in $\mathcal{R}$, and is called *free-paired-edge* if both of its vertices are not in $\mathcal{R}$.

Definition 2. A matched-paired-dominating set MPD of G w.r.t. $\mathcal{R}$ is called *(k, s, f)-matched-paired-dominating set* if (1) $|MPD| = k+s+f$; (2) there are exactly k full-paired-edges in MPD; (3) there are exactly s semi-paired-edges in MPD, and (4) all other paired-edges in MPD are free-paired-edges.

By the above definition, a paired-edge in a (k, s, f)-matched-paired-dominating set MPD is either a full-paired-edge, a semi-paired-edge or a free-paired-edge. Then, the matched number of MPD is $|V(MPD) \cap \mathcal{R}| = 2k+s$. Thus, a maximum (k^*, s^*, f^*)-matched-paired-dominating set of a graph G satisfies that $\beta(G) = 2k^* + s^* \geqslant 2k + s$ for any (k, s, f)-matched-paired-dominating set of G.

Definition 3. Let MPD be a (k, s, f)-matched-paired-dominating set of a graph G w.r.t. $\mathcal{R}$. Define $K_G(MPD)$, $S_G(MPD)$, and $F_G(MPD)$ to be the subsets of MPD consisting of all full-paired-edges, all semi-paired-edges, and all free-paired-edges in MPD, respectively.

For example, let G be a graph with restricted vertex set $\mathcal{R} = \{v_2, v_3\}$ shown in Fig. 2. Let $MPD_1 = \{\langle v_2, v_3 \rangle, \langle v_1, v_5 \rangle\}$ and let $MPD_2 = \{\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle\}$. Then, MPD_1 is a $(1, 0, 1)$-matched-paired-dominating set and MPD_2 is a $(0, 2, 0)$-matched-paired-dominating set. By definition, $K_G(MPD_1) = \{\langle v_2, v_3 \rangle\}$, $S_G(MPD_1) = \emptyset$, and $F_G(MPD_1) = \{\langle v_1, v_5 \rangle\}$, where $|K_G(MPD_1)| = 1$, $|S_G(MPD_1)| = 0$, and $|F_G(MPD_1)| = 1$.

Next, we introduce cographs. A graph is a cograph if there is no induced path containing four vertices [9]. Such graphs are exactly the class of distance-hereditary graphs with diameters less than or equal to two [1]. Every cograph can be recursively defined as follows.

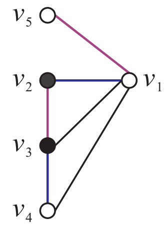


Fig. 2: A graph G with restricted vertex set $\mathcal{R} = \{v_2, v_3\}$, where restricted vertices are drawn by filled circles.

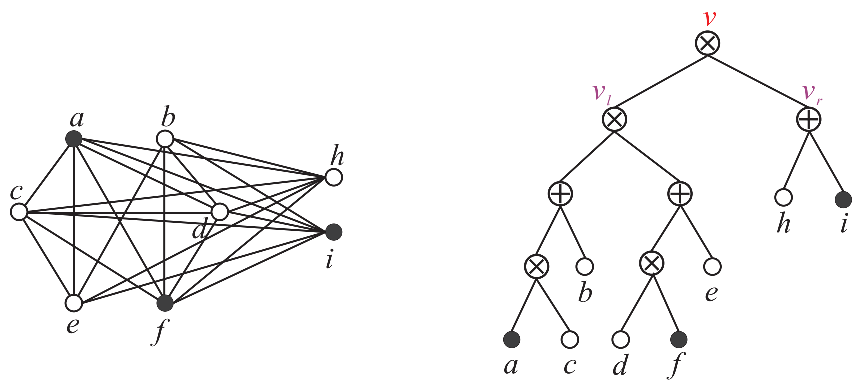
Definition 4. [9, 10] The class of cographs can be defined by the following recursive definition:
(1) A graph consisting of a single vertex and no edges is a cograph.
(2) If $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ are cographs, then the *union* G of G_L and G_R, denoted by $G = G_L \oplus G_R = (V_L \cup V_R, E_L \cup E_R)$, is a cograph. In this case, we say that G is formed from G_L and G_R by a *union operation*.
(3) If $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ are cographs, then the *joint* G of G_L and G_R, denoted by $G = G_L \otimes G_R = (V_L \cup V_R, E_L \cup E_R \cup \widehat{E})$, is a cograph, where $\widehat{E} = \{uv | \forall u \in V_L \text{ and } \forall v \in V_R\}$. In this case, we say that G is formed from G_L and G_R by a *joint operation*.

A cograph G can be represented by a rooted binary tree $DT(G)$, called a *decomposition tree* [4, 9]. The leaf nodes of $DT(G)$ represent the vertices of G. Each internal node of $DT(G)$ is labeled by either '$\oplus$' or '$\otimes$'. The cograph corresponding to a $\oplus$-labeled (resp. $\otimes$-labeled) node v in $DT(G)$ is obtained from the cographs corresponding to the children of v in $DT(G)$ by means of a *union* (resp. *joint*) operation. A decomposition tree of a cograph can be constructed as follows.

Definition 5. [4] The decomposition tree $DT(G)$ of a cograph G consisting of a single vertex v is a tree of one node labeled by v. If G is formed from G_L and G_R by a union (resp. joint) operation, then the root of the decomposition $DT(G)$ is a node labeled by $\oplus$ (resp. $\otimes$) with the roots of $DT(G_L)$ and $DT(G_R)$ being the children of the root of $DT(G)$, respectively.

The decomposition tree $DT(G)$ of a cograph G is a rooted and unordered binary tree. Note that exchanging the left and right children of an internal node in $DT(G)$ will be also a decomposition tree of G. For instance, given a cograph G shown in Fig. 3(a), the decomposition tree $DT(G)$ of G is shown in Fig. 3(b).

Theorem 4. *[4, 9] A decomposition tree $DT(G)$ of a cograph $G = (V, E)$ can be constructed in $O(|V| + |E|)$-linear time.*



(a) (b)

Fig. 3: (a) A cograph G with restricted vertex set $\mathcal{R} = \{a, f, i\}$, and (b) a decomposition tree $DT(G)$ of G, where restricted vertices are drawn by filled circles.

3 The Maximum Matched-Paired-Domination Problem on Cographs

In this section, we will show that the maximum matched-paired-domination problem on cographs is linear solvable. Recall that a canonical matched-paired-dominating set of a graph is a maximum matched-paired-dominating set with the least free-paired-edges. In fact, we will construct a canonical matched-paired-dominating set of a connected cograph in linear time. We first give the following lemma to show some properties of a maximum matched-paired-dominating set of a graph.

Lemma 5. *Assume G is a connected graph without isolated vertices and $\mathcal{R}$ is a restricted vertex set of G. Let $MMPD$ be a maximum matched-paired-dominating set of G w.r.t. $\mathcal{R}$ and let $\upsilon \in \mathcal{R} - V(MMPD)$. Then, the following statements hold true:*
(1) if $\langle v_f, \widetilde{v}_f \rangle$ is a free-paired-edge in $MMPD$, then υ is adjacent to neither v_f nor $\widetilde{v}_f$;
(2) $N_G(\upsilon) \subseteq V(MMPD)$;
(3) if $\langle v_x, \widetilde{v}_x \rangle$ is a semi-paired-edge or full-paired-edge in $MMPD$ and υ is adjacent to v_x, then $N_G(\widetilde{v}_x) - \{\upsilon\} \subseteq V(MMPD)$;
(4) if $\langle v_f, v_r \rangle$ is a semi-paired-edge, with restricted vertex v_r, in $MMPD$, then υ is not adjacent to v_r.

Proof. We first prove Statement (1). Assume by contradiction that υ is adjacent to v_f. If $N_G(\widetilde{v}_f) - V(MMPD) = \emptyset$, then $MMPD - \{\langle v_f, \widetilde{v}_f \rangle\} \cup \{\langle v_f, \upsilon \rangle\}$ is a matched-paired-dominating set of G having more restricted vertices than $MMPD$, a contradiction. Consider $N_G(\widetilde{v}_f) - V(MMPD) \neq \emptyset$. Let $\widetilde{\upsilon} \in N_G(\widetilde{v}_f) - V(MMPD)$. Then, $MMPD - \{\langle v_f, \widetilde{v}_f \rangle\} \cup \{\langle v_f, \upsilon \rangle, \langle \widetilde{v}_f, \widetilde{\upsilon} \rangle\}$

is a matched-paired-dominating set of G having more restricted vertices than $MMPD$, a contradiction. Thus, v is not adjacent to v_f and Statement (1) holds true. Statement (2) is clearly true. Otherwise, $MMPD \cup \{\langle v, \widetilde{v}\rangle\}$, where $\widetilde{v} \in N_G(v) - V(MMPD)$, is a matched-paired-dominating set of G which has more restricted vertices than $MMPD$, a contradiction.

Next, we prove Statement (3). Assume by contradiction that $N_G(\widetilde{v}_x) - \{v\} \nsubseteq V(MMPD)$. Let $\widetilde{v} \in (N_G(\widetilde{v}_x) - \{v\}) - V(MMPD)$. By Statement (2), $N_G(v) \subseteq V(MMPD)$. Thus, $\widetilde{v} \notin N_G(v)$. Then, $MMPD - \{\langle v_x, \widetilde{v}_x\rangle\} \cup \{\langle \widetilde{v}_x, \widetilde{v}\rangle, \langle v_x, v\rangle\}$ is a matched-paired-dominating set of G having more restricted vertices than $MMPD$, a contradiction. Thus, $N_G(\widetilde{v}_x) - \{v\} \subseteq V(MMPD)$.

Finally, we prove Statement (4). Assume by contradiction that v is adjacent to v_r. By Statement (3), $N_G(v_f) - \{v\} \subseteq V(MMPD)$. Then, v_f is dominated by one vertex of $N_G(v_f) \cap V(MMPD)$, e.g. v_r. Thus, $MMPD - \{\langle v_f, v_r\rangle\} \cup \{\langle v_r, v\rangle\}$ is a matched-paired-dominating set of G having more restricted vertices than $MMPD$, a contradiction. Thus, v is not adjacent to v_r. $\square$

By Theorem 4, a decomposition tree $DT(G)$ of a cograph $G = (V, E)$ can be constructed in $O(|V| + |E|)$-linear time. A cograph is not connected if the root of its corresponding decomposition tree is a $\oplus$-labeled node. Hence, we assume that the root of the corresponding decomposition tree is a $\otimes$-labeled node. For a node v in $DT(G)$, denote by $DT_v(G)$ the subtree of $DT(G)$ rooted at v, and denote by G_v the subgraph of G induced by the leaves of $DT_v(G)$. Our algorithm is sketched as follows: The algorithm is given a decomposition tree $DT(G)$ of a cograph G and a restricted vertex set $\mathcal{R}$ in G. It visits nodes of $DT(G)$ in a postorder sequence (i.e., bottom-up manner). Thus, while visiting a node, both its children were visited. Suppose that it is about to process internal node v with v_l and v_r being the left and right children of v in $DT(G)$, respectively. Let $\mathcal{R}_L$ and $\mathcal{R}_R$ be the restricted vertex sets of G_{v_l} and G_{v_r}, respectively, such that $|\mathcal{R}_L| \geqslant |\mathcal{R}_R|$. Let $CMPD_L$ be a canonical matched-paired-dominating set of G_{v_l} and let V_R be the vertex set of G_{v_r}. Then, it uses $CMPD_L$ and V_R to construct a canonical matched-paired-dominating set $CMPD$ of G_v. If v is the root of $DT(G)$, then $CMPD$ is a canonical matched-paired-dominating set of G and the algorithm terminates. For example, let G be cograph with restricted vertex set $\mathcal{R} = \{a, f, i\}$ shown in Fig. 3. Our algorithm traverses the decomposition tree $DT(G)$ in a bottom-up manner. Suppose that it is about to process the root v with v_l and v_r being the left and right children of v in $DT(G)$, respectively. Then, a canonical $(1,0,0)$-matched-paired-dominating set $CMPD_L = \{\langle a, f\rangle\}$ of G_{v_l} and the vertex set $V_R = \{h, i\}$ of G_{v_r} have been computed. The algorithm then constructs from $CMPD_L$ and V_R a canonical $(1,1,0)$-matched-paired-dominating set $\{\langle a, f\rangle, \langle b, i\rangle\}$ of G_v. In the following, we

will show how to construct such a canonical matched-paired-dominating set.

In the rest of the paper, we assume that $G = (V, E)$ is a cograph with restricted vertex set $\mathcal{R}$ and is formed from G_L and G_R by either a union operation or a joint operation. We use V_L and V_R to denote the vertex sets of G_L and G_R, respectively. In other words, $V = V_L \cup V_R$ and $V_L \cap V_R = \emptyset$. Notice that every vertex in V_L is adjacent to all vertices in V_R if $G = G_L \otimes G_R$. On the other hand, we use $\mathcal{R}_L$ and $\mathcal{R}_R$ to denote the restricted vertex sets of G_L and G_R, respectively, i.e., $\mathcal{R}_L = \mathcal{R} \cap V_L$ and $\mathcal{R}_R = \mathcal{R} \cap V_R$.

By the definition of cographs, G_L or G_R may contain isolated vertices. For a graph H, we use $I(H)$ to denote the set of isolated vertices in H. We denote by $H - I(H)$ deleting $I(H)$ from H. Then, $I(G_L)$ and $I(G_R)$ are the sets of isolated vertices in G_L and G_R, respectively. By Corollary 2, $G_L - I(G_L)$ and $G_R - I(G_R)$ have matched-paired-dominating sets if they are not empty, and, hence, they have canonical matched-paired-dominating sets. Then, the following lemma can be easily verified from the definition of union operation.

Lemma 6. *Assume $G = G_L \oplus G_R$ is a cograph with restricted vertex set $\mathcal{R}$. Let $CMPD_L$ and $CMPD_R$ be canonical matched-paired-dominating sets of $G_L - I(G_L)$ and $G_R - I(G_R)$ w.r.t. $\mathcal{R}_L - I(G_L)$ and $\mathcal{R}_R - I(G_R)$, respectively. Then, $I(G) = I(G_L) \cup I(G_R)$ and $CMPD_L \cup CMPD_R$ is a canonical matched-paired-dominating set of $G - I(G)$ w.r.t. $\mathcal{R} - I(G)$.*

From now on, we consider that G is formed from G_L and G_R by a joint operation. First, we consider that $\mathcal{R} = \emptyset$. Let $v_L \in V_L$ and let $v_R \in V_R$. Obviously, $CMPD = \{\langle v_L, v_R \rangle\}$ is a matched-paired-dominating set of G, and, hence, the the maximum matched-paired-domination problem on G is trivially solvable. In the following, we assume $\mathcal{R} \neq \emptyset$. For the case of $|\mathcal{R}_L| = |\mathcal{R}_R|$, we give the following lemma to find a canonical matched-paired-dominating set of G.

Lemma 7. *Assume $G = G_L \otimes G_R$ is a cograph with restricted vertex set $\mathcal{R}$, $\mathcal{R}_L = \mathcal{R} \cap V_L$, and $\mathcal{R}_R = \mathcal{R} \cap V_R$. If $|\mathcal{R}_L| = |\mathcal{R}_R|$ and $|\mathcal{R}_L| > 0$, then there exists a canonical matched-paired-dominating set $CMPD$ of G w.r.t. $\mathcal{R}$ such that $V(CMPD) = \mathcal{R}$, $|CMPD| = \frac{|\mathcal{R}|}{2}$, and $CMPD$ contains no free-paired-edge.*

Proof. Let $\mathcal{R}_L = \{u_1, u_2, \cdots, u_k\}$ and $\mathcal{R}_R = \{v_1, v_2, \cdots, v_k\}$, where $k = \frac{|\mathcal{R}|}{2}$. By pairing u_i with v_i for $1 \leqslant i \leqslant k$, we obtain a $(k, 0, 0)$-matched-paired-dominating set $CMPD = \{\langle u_1, v_1 \rangle, \langle u_2, v_2 \rangle, \cdots, \langle u_k, v_k \rangle\}$ of G with cardinality $\frac{|\mathcal{R}|}{2}$. By Lemma 3, $CMPD$ is a canonical matched-paired-dominating set of G w.r.t. $\mathcal{R}$ without free-paired-edges. $\square$

From now on, we assume that $|\mathcal{R}_L| \neq |\mathcal{R}_R|$. Without loss of generality, assume $|\mathcal{R}_L| > |\mathcal{R}_R|$. Let $CMPD_L$ be a canonical (k_L, s_L, f_L)-matched-paired-dominating set of $G_L - I(G_L)$ w.r.t.

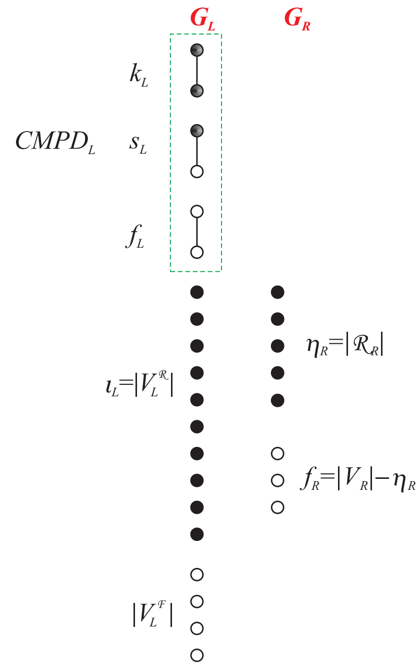


Fig. 4: The partition of V_L and V_R.

$\mathcal{R}_L - I(G_L)$. We first partition $V_L - V(CMPD_L)$ into two subsets $V_L^{\mathcal{R}}$ and $V_L^{\mathcal{F}}$ such that $V_L^{\mathcal{R}} = \mathcal{R}_L - V(CMPD_L)$ and $V_L^{\mathcal{F}} \cap \mathcal{R}_L = \emptyset$. Note that $V_L^{\mathcal{R}}$ or $V_L^{\mathcal{F}}$ may contain isolated vertices of G_L. By Statement (2) of Lemma 5, $N_{G_L}(v) \subseteq V(CMPD_L)$ for $v \in V_L^{\mathcal{R}} - I(G_L)$. We next partition V_R into two subsets $\mathcal{R}_R$ and $V_R - \mathcal{R}_R$. The partition of V_L and V_R is shown in Fig. 4. For simplicity, let $\imath_L = |V_L^{\mathcal{R}}|$, $\eta_R = |\mathcal{R}_R|$, and let $f_R = |V_R| - \eta_R$. By definition, $|\mathcal{R}_L| = 2k_L + s_L + \imath_L$ and $|\mathcal{R}_R| = \eta_R$. By assumption, $|\mathcal{R}_L| > |\mathcal{R}_R|$. Thus, we get that

$$2k_L + s_L + \imath_L > \eta_R. \tag{1}$$

Considering the relation between $\imath_L$ and $\eta_R + f_R$, we have that $\imath_L \geqslant \eta_R + f_R$ or $\imath_L < \eta_R + f_R$. We construct from $CMPD_L$ and V_R a matched-paired-dominating set $CMPD$ of G having at most one free-paired-edge as follows:

Case 1: $\imath_L \geqslant \eta_R + f_R$. Let $V_L' = \{u_1, u_2, \cdots, u_{|V_L'|}\}$ be a subset of $V_L^{\mathcal{R}}$ such that $|V_L'| = \eta_R + f_R = |V_R|$, and let $V_R = \{v_1, v_2, \cdots, v_{|V_R|}\}$. By pairing u_i with v_i for $1 \leqslant i \leqslant \eta_R + f_R$, we construct a $(k_L + \eta_R, s_L + f_R, 0)$-matched-paired-dominating set $CMPD = K_{G_L}(CMPD_L) \cup S_{G_L}(CMPD_L) \cup_{1 \leqslant i \leqslant \eta_R + f_R} \{\langle u_i, v_i \rangle\}$.

Case 2: $\imath_L < \eta_R + f_R$. There are three subcases:

Case 2.1: $\imath_L > \eta_R$. In this subcase, $\eta_R < \imath_L < \eta_R + f_R$. Thus, $0 < \imath_L - \eta_R < f_R$. Let $V_L^{\mathcal{R}} = \{u_1, u_2, \cdots, u_{\imath_L}\}$, $\mathcal{R}_R = \{v_1, v_2, \cdots, v_{\eta_R}\}$, and let $V_R' = \{v_{\eta_R+1}, v_{\eta_R+2}, \cdots, v_{\imath_L}\}$

be a subset of $V_R - \mathcal{R}_R$ with $|V'_R| = \imath_L - \eta_R$. By pairing u_i with v_i for $1 \leqslant i \leqslant \imath_L$, we obtain a $(k_L + \eta_R, s_L + \imath_L - \eta_R, 0)$-matched-paired-dominating set $CMPD = K_{G_L}(CMPD_L) \cup S_{G_L}(CMPD_L) \cup_{1 \leqslant i \leqslant \imath_L} \{\langle u_i, v_i \rangle\}$. Then, $V(CMPD) \cap \mathcal{R} = \mathcal{R}$ and $CMPD$ contains no free-paired-edge. Thus, $CMPD$ is a canonical matched-paired-dominating set of G. Fig. 5(a) depicts the construction of $CMPD$ in the subcase.

Case 2.2: $\imath_L < \eta_R$. By Eq. (1), $2k_L + s_L + \imath_L > \eta_R$. Let $\eta'_R = \eta_R - \imath_L$. Then, $2k_L + s_L > \eta_R - \imath_L = \eta'_R > 0$. We partition $\mathcal{R}_R$ into two subsets $\mathcal{R}_R^\alpha$ and $\mathcal{R}_R^\beta$ such that $|\mathcal{R}_R^\alpha| = \imath_L$ and $|\mathcal{R}_R^\beta| = \eta'_R = \eta_R - \imath_L$. By pairing every vertex in $\mathcal{R}_R^\alpha$ with a vertex in $V_L^{\mathcal{R}}$, we obtain a set $\mathcal{K}$ of $\imath_L$ full-paired-edges shown in Fig. 5(b). We then consider the following two subcases:

Case 2.2.1: $\eta_R - \imath_L = \eta'_R \leqslant s_L$. We first partition $S_{G_L}(CMPD_L)$ into two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $|\mathcal{S}_1| = \eta'_R$ and $|\mathcal{S}_2| = s_L - \eta'_R$. Let $u_1, u_2, \cdots, u_{\eta'_R}$ be the restricted vertices in $V(\mathcal{S}_1)$ and let $\mathcal{R}_R^\beta = \{v_1, v_2, \cdots, v_{\eta'_R}\}$. By pairing u_i with v_i for $1 \leqslant i \leqslant \eta'_R$, we obtain a set $\mathcal{K}_1$ of η'_R full-paired-edges. Let $CMPD = K_{G_L}(CMPD_L) \cup \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{S}_2$. Then, $CMPD$ is a $(k_L + \eta_R, s_L - \eta'_R, 0)$-matched-paired-dominating set of G. Since $V(CMPD) \cap \mathcal{R} = \mathcal{R}$, $CMPD$ is a maximum matched-paired-dominating set of G. Thus, $CMPD$ is a canonical matched-paired-dominating set of G. The construction of $CMPD$ is shown in Fig. 5(c).

Case 2.2.2: $\eta_R - \imath_L = \eta'_R > s_L$. We first partition $\mathcal{R}_R^\beta$ into two subsets $\mathcal{R}_R^{\mathrm{a}}$ and $\mathcal{R}_R^{\mathrm{b}}$ such that $|\mathcal{R}_R^{\mathrm{a}}| = s_L$ and $|\mathcal{R}_R^{\mathrm{b}}| = \eta'_R - s_L$. Let $u_1, u_2, \cdots, u_{s_L}$ be the restricted vertices in $V(S_{G_L}(CMPD_L))$ and let $\mathcal{R}_R^{\mathrm{a}} = \{v_1, v_2, \cdots, v_{s_L}\}$. By pairing u_i with v_i for $1 \leqslant i \leqslant s_L$, we obtain a set $\mathcal{K}_1$ of s_L full-paired-edges. Suppose that $|\mathcal{R}_R^{\mathrm{b}}| = \eta'_R - s_L$ is even. We first partition $K_{G_L}(CMPD_L)$ into two subsets $\mathcal{K}_{L_1}$ and $\mathcal{K}_{L_2}$ such that $\mathcal{K}_{L_1}$ contains $\frac{|\mathcal{R}_R^{\mathrm{b}}|}{2}$ full-paired-edges, i.e., $V(\mathcal{K}_{L_1})$ contains $|\mathcal{R}_R^{\mathrm{b}}|$ restricted vertices. By pairing every vertex of $V(\mathcal{K}_{L_1})$ with a vertex in $\mathcal{R}_R^{\mathrm{b}}$, we obtain a set $\mathcal{K}_2$ of $|\mathcal{R}_R^{\mathrm{b}}|$ full-paired-edges. Let $CMPD = \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_{L_2}$. Then, $CMPD$ is a $(k_L + \frac{\imath_L + s_L + \eta_R}{2}, 0, 0)$-matched-paired-dominating set of G. We can see that $CMPD$ is a $(\frac{|\mathcal{R}|}{2}, 0, 0)$-matched-paired-dominating set of G, $V(CMPD) \cap \mathcal{R} = \mathcal{R}$, and that $CMPD$ contains no free-paired-edge. On the other hand, suppose that $|\mathcal{R}_R^{\mathrm{b}}| = \eta'_R - s_L$ is odd. Then, $|\mathcal{R}|$ is odd. We pick from G a restricted vertex $\widetilde{v}$ and a free vertex v_f to form a semi-paired-edge as follows: Case i, $V_L - \mathcal{R}_L \neq \emptyset$. Let $\widetilde{v} \in \mathcal{R}_R^{\mathrm{b}}$ and let $v_f \in V_L - \mathcal{R}_L$. Case ii, $V_L - \mathcal{R}_L = \emptyset$ and $f_R > |I(G_R) - \mathcal{R}_R|$. Let $\widetilde{v}$ be a restricted vertex in $\mathcal{R}_R^{\mathrm{b}}$ such that $\widetilde{v}$ is adjacent to one free vertex v_f in $V_R - (\mathcal{R}_R \cup I(G_R))$. Case iii, $V_L - \mathcal{R}_L = \emptyset$ and $f_R = |I(G_R) - \mathcal{R}_R| \neq 0$. Let $\langle \widetilde{v}, v_L \rangle$ be a full-paired-edge in $K_{G_L}(CMPD_L)$ and let v_f be a free vertex in $I(G_R) - \mathcal{R}_R$. For case of $V_L - \mathcal{R}_L = \emptyset$ and $f_R = |I(G_R) - \mathcal{R}_R| = 0$, we have that $|V| = |\mathcal{R}|$ is odd and a $(\lfloor \frac{|\mathcal{R}|}{2} \rfloor, 0, 0)$-matched-paired-dominating set $CMPD$ of G can be easily constructed from $K_{G_L}(CMPD_L)$ and $\mathcal{R}_R$. By Statement (2) of Lemma 3, $CMPD$ is a canonical matched-paired-dominating set of G.

Now, suppose $\widetilde{v}$ and v_f exist. Let $\mathcal{S} = \{\langle \widetilde{v}, v_f \rangle\}$. If $\widetilde{v} \notin \mathcal{R}_L$, then let $\widetilde{\mathcal{K}} = \emptyset$ and $\mathcal{R}_R^{\rm b} = \mathcal{R}_R^{\rm b} - \{\widetilde{v}\}$; otherwise, let $\langle \widetilde{v}, v_L \rangle \in K_{G_L}(CMPD_L)$, $K_{G_L}(CMPD_L) = K_{G_L}(CMPD_L) - \{\langle \widetilde{v}, v_L \rangle\}$, $v_R \in \mathcal{R}_R^{\rm b}$, $\mathcal{R}_R^{\rm b} = \mathcal{R}_R^{\rm b} - \{v_R\}$, and let $\widetilde{\mathcal{K}} = \{\langle v_L, v_R \rangle\}$. Then, $|\mathcal{R}_R^{\rm b}|$ becomes even. We then partition $K_{G_L}(CMPD_L)$ into two subsets $\mathcal{K}_{L_1}$ and $\mathcal{K}_{L_2}$ such that $\mathcal{K}_{L_1}$ contains $\frac{|\mathcal{R}_R^{\rm b}|}{2}$ full-paired-edges. By pairing every vertex of $V(\mathcal{K}_{L_1})$ with a vertex in $\mathcal{R}_R^{\rm b}$, we obtain a set $\mathcal{K}_2$ of $|\mathcal{R}_R^{\rm b}|$ full-paired-edges. Let $CMPD = \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_{L_2} \cup \widetilde{\mathcal{K}} \cup \mathcal{S}$. Then, $CMPD$ is a $(k_L + \lfloor \frac{\imath_L + s_L + \eta_R}{2} \rfloor, 1, 0)$-matched-paired-dominating set of G. We can see that $CMPD$ is a $(\lfloor \frac{|\mathcal{R}|}{2} \rfloor, 1, 0)$-matched-paired-dominating set of G. By Statement (1) of Lemma 3, $CMPD$ is a canonical matched-paired-dominating set of G. The construction of $CMPD$ is shown in Fig. 5(d).

Case 2.3: $\imath_L = \eta_R$. In this subcase, $\imath_L = \eta_R < \eta_R + f_R$. Consider the following two subcases:

Case 2.3.1: $\imath_L \neq 0$. Let $V_L^{\mathcal{R}} = \{u_1, u_2, \cdots, u_{\imath_L}\}$ be the restricted vertex set of $V_L - V(CMPD_L)$ and let $\mathcal{R}_R = \{v_1, v_2, \cdots, v_{\imath_L}\}$. By paring u_i with v_i for $1 \leqslant i \leqslant \imath_L$, we get a set $\mathcal{K}$ of $\imath_L$ full-paired-edges. Let $CMPD = K_{G_L}(CMPD_L) \cup S_{G_L}(CMPD_L) \cup \mathcal{K}$. Then, $CMPD$ is a $(k_L + \imath_L, s_L, 0)$-matched-paired-dominating set of G. We can see that $CMPD$ is a maximum matched-paired-dominating set of G without free-paired-edges. Thus, $CMPD$ is a canonical matched-paired-dominating set of G. Fig. 6(b) shows the construction of $CMPD$ in this subcase.

Case 2.3.2: $\imath_L = 0$. First, we consider that $s_L \neq 0$. Let $\langle v_L, v_f \rangle$ be a semi-paired-edge, with restricted vertex v_L, in $S_{G_L}(CMPD_L)$, and let v_R be a free vertex in V_R. Let $\mathcal{S} = \{\langle v_L, v_R \rangle\}$. Then, $CMPD = K_{G_L}(CMPD_L) \cup S_{G_L}(CMPD_L) - \{\langle v_L, v_f \rangle\} \cup \mathcal{S}$ is a $(k_L, s_L, 0)$-matched-paired-dominating set of G such that $V(CMPD) \cap \mathcal{R} = \mathcal{R}$. It is easy to see that $CMPD$ is a canonical matched-paired-dominating set of G. Fig. 6(c) shows the construction of $CMPD$ in case of $\imath_L = \eta_R = 0$ and $s_L \neq 0$. On the other hand, we consider that $s_L = 0$. If $V(K_{G_L}(CMPD_L))$ is a dominating set of G_L, i.e., $f_L = 0$ and $I(G_L) = \emptyset$, then $CMPD = K_{G_L}(CMPD_L)$ is clearly a canonical matched-paired-dominating set of G. Suppose that $f_L \neq 0$ or $I(G_L) \neq \emptyset$. Consider that $f_R \geqslant 2$. Let v_{f_1} and v_{f_2} be two free vertices in V_R, and let $\langle v_{r_1}, v_{r_2} \rangle$ be a full-paired-edge in $K_{G_L}(CMPD_L)$. Then, $CMPD = K_{G_L}(CMPD_L) - \{\langle v_{r_1}, v_{r_2} \rangle\} \cup \{\langle v_{f_1}, v_{r_1} \rangle, \langle v_{f_2}, v_{r_2} \rangle\}$ is a canonical $(k_L - 1, 2, 0)$-matched-paired-dominating set of G with that $V(CMPD) \cap \mathcal{R} = \mathcal{R}$. Next, consider that $f_R = 1$. Let v_{R_f} be the only vertex in V_R. Consider the following cases: Case i, there exists one restricted vertex $\widetilde{v}_L$ in $\mathcal{R}_L$ such that $N_{G_L}(\widetilde{v}_L) \nsubseteq \mathcal{R}_L$. Let $v_{L_f} \in N_{G_L}(\widetilde{v}_L) - \mathcal{R}_L$ and let $\langle \widetilde{v}_L, v_L \rangle$ be a full-paired-edge in $K_{G_L}(CMPD_L)$. Let $CMPD = K_{G_L}(CMPD_L) - \{\langle \widetilde{v}_L, v_L \rangle\} \cup \{\langle v_L, v_{R_f} \rangle, \langle \widetilde{v}_L, v_{L_f} \rangle\}$. Then, $CMPD$ is a canonical $(k_L - 1, 2, 0)$-matched-paired-dominating set of G with that $V(CMPD) \cap \mathcal{R} = \mathcal{R}$. Case ii, $N_{G_L}(\widetilde{v}_L) \subseteq \mathcal{R}_L$

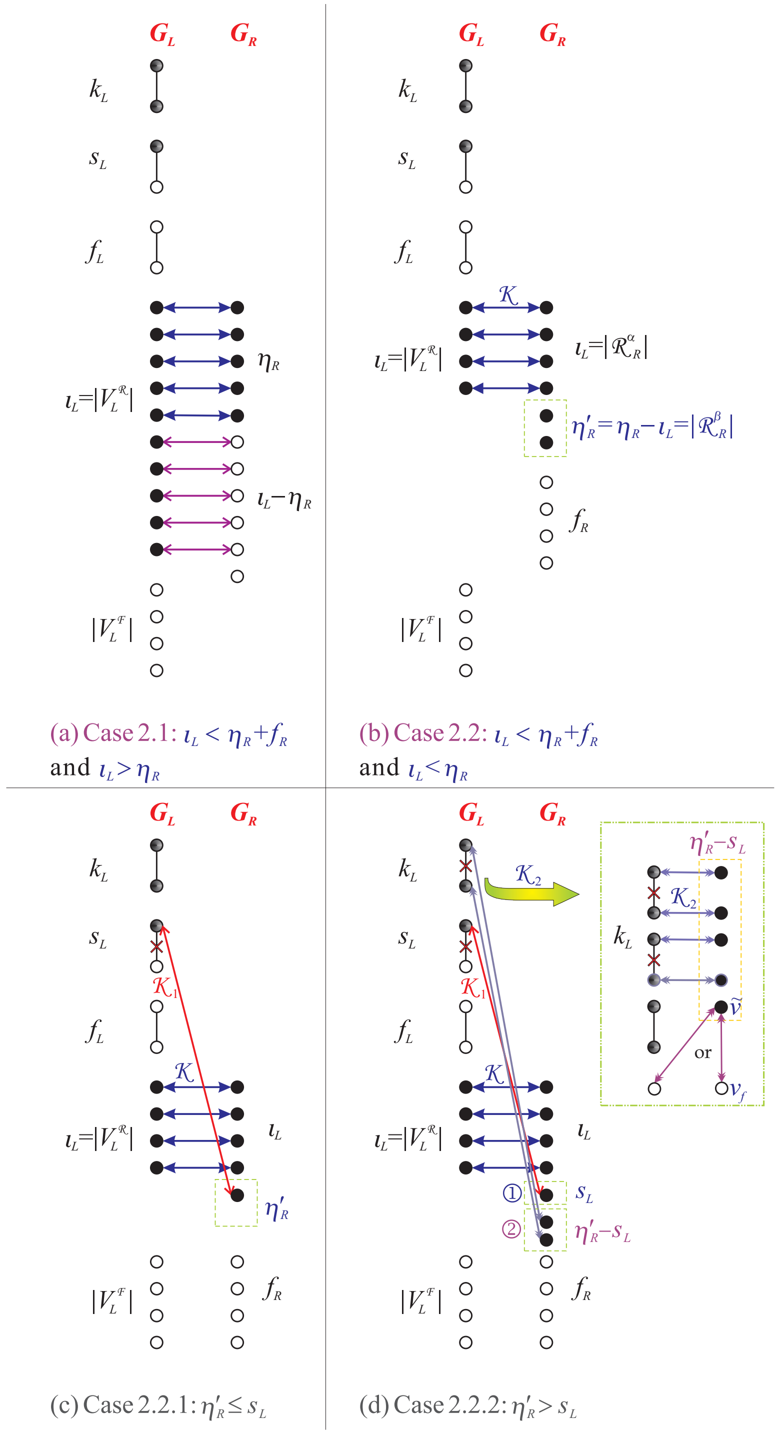


Fig. 5: The construction of a matched-paired-dominating set $CMPD$ of G for (a) Case 2.1, and (b)–(d) Case 2.2, where restricted vertices are drawn by filled circles, symbol '$\times$' denotes the destruction to one paired-edge in $CMPD_L$, and arrow lines represent the new paired-edges for the construction.

for each $\widetilde{v}_L \in \mathcal{R}_L$. Let $v_{L_f} \in V_L - \mathcal{R}_L$ and let $CMPD = K_{G_L}(CMPD_L) \cup \{\langle v_{L_f}, v_{R_f}\rangle\}$. Then, $CMPD$ is a $(k_L, 0, 1)$-matched-paired-dominating set of G. We can see that if $N_{G_L}(\widetilde{v}_L) \subseteq \mathcal{R}_L$ for each $\widetilde{v}_L \in \mathcal{R}_L$, then a free-paired-edge is necessary for constructing a maximum matched-paired-dominating set of G. Thus, $CMPD$ is a canonical matched-paired-dominating set of G. Fig. 6(d) depicts the construction of $CMPD$ in case of $\imath_L = \eta_R = 0$ and $s_L = 0$. $\square$

It follows from the above constructions and arguments that our constructed matched-paired-dominating set $CMPD$ for case of $\imath_L < \eta_R + f_R$ (Case 2) is a canonical matched-paired-dominating set of G. The remnant is to prove that the constructed matched-paired-dominating set $CMPD$ for case of $\imath_L \geqslant \eta_R + f_R$ (Case 1) is a canonical matched-paired-dominating set of G. The following lemma shows the result.

Lemma 8. *Assume $G = G_L \otimes G_R$ is a cograph with restricted vertex set $\mathcal{R}$, $\mathcal{R}_L = \mathcal{R} \cap V_L$, $\mathcal{R}_R = \mathcal{R} \cap V_R$, and $|\mathcal{R}_L| > |\mathcal{R}_R|$. Let $CMPD_L$ be a canonical (k_L, s_L, f_L)-matched-paired-dominating set of $G_L - I(G_L)$, $\imath_L = |\mathcal{R}_L - V(CMPD_L)|$, $\eta_R = |\mathcal{R}_R|$, and let $f_R = |V_R| - \eta_R$. If $\imath_L \geqslant \eta_R + f_R$, then the constructed $(k_L + \eta_R, s_L + f_R, 0)$-matched-paired-dominating set $CMPD$ is a canonical matched-paired-dominating set of G.*

Proof. In case of $\imath_L \geqslant \eta_R + f_R$, the construction of $CMPD$ is shown in Fig. 7(a). A paired-edge in a matched-paired-dominating set of G is called *mixed* if one of its vertices is in V_L and the other is in V_R. Suppose that $MMPD$ is a maximum matched-paired-dominating set of G with the least free-paired-edges. That is, $MMPD$ is a canonical matched-paired-dominating set of G. We may assume that $MMPD$ is chosen such that the number of mixed paired-edges is maximal. Denote by $MMPD_{|G_L}$ (resp. $MMPD_{|G_R}$) a restriction of $MMPD$ to G_L (resp. G_R). The set of mixed paired-edges of $MMPD$ is partitioned into four subsets K, S_1, S_2, F such that K contains all mixed full-paired-edges, S_1 contains all mixed semi-paired-edges with restricted vertices being in V_L, S_2 contains all mixed semi-paired-edges with restricted vertices being in V_R, and F contains all mixed free-paired-edges. The set of paired-edges of $MMPD_{|G_L}$ (resp. $MMPD_{|G_R}$) is partitioned into three subsets K_L, S_L, F_L (resp. K_R, S_R, and F_R) containing full-paired-edges, semi-paired-edges, and free-paired-edges, respectively. Let $I_L = \mathcal{R}_L - V(MMPD)$ and $I_R = \mathcal{R}_R - V(MMPD)$. For simplicity, let $|K| = k$, $|S_1| = s_1$, $|S_2| = s_2$, $|F| = f$, $|K_L| = k'_L$, $|S_L| = s'_L$, $|F_L| = f'_L$, $|K_R| = k'_R$, $|S_R| = s'_R$, $|F_R| = f'_R$, $|I_L| = i'_L$, and let $|I_R| = i'_R$. The possible paired-edges in $MMPD$ are shown in Fig. 7(b). Since $|\mathcal{R}_L| > |\mathcal{R}_R|$ and $MMPD$ is a canonical matched-paired-dominating set of G, $f \leqslant 1$ and at least one of i'_L and i'_R equals to 0.

We first prove Claim (1) that $2k_L + s_L \geqslant 2k'_L + s'_L$. We prove it by constructing from $MMPD_{|G_L}$ a matched-paired-dominating set $MMPD_L$ of $G_L - I(G_L)$ such that $|V(MMPD_L) \cap (\mathcal{R}_L - I(G_L))| \geqslant 2k'_L + s'_L$. The construction is as follows: Initially, let $MMPD_L = K_L \cup S_L$. Let

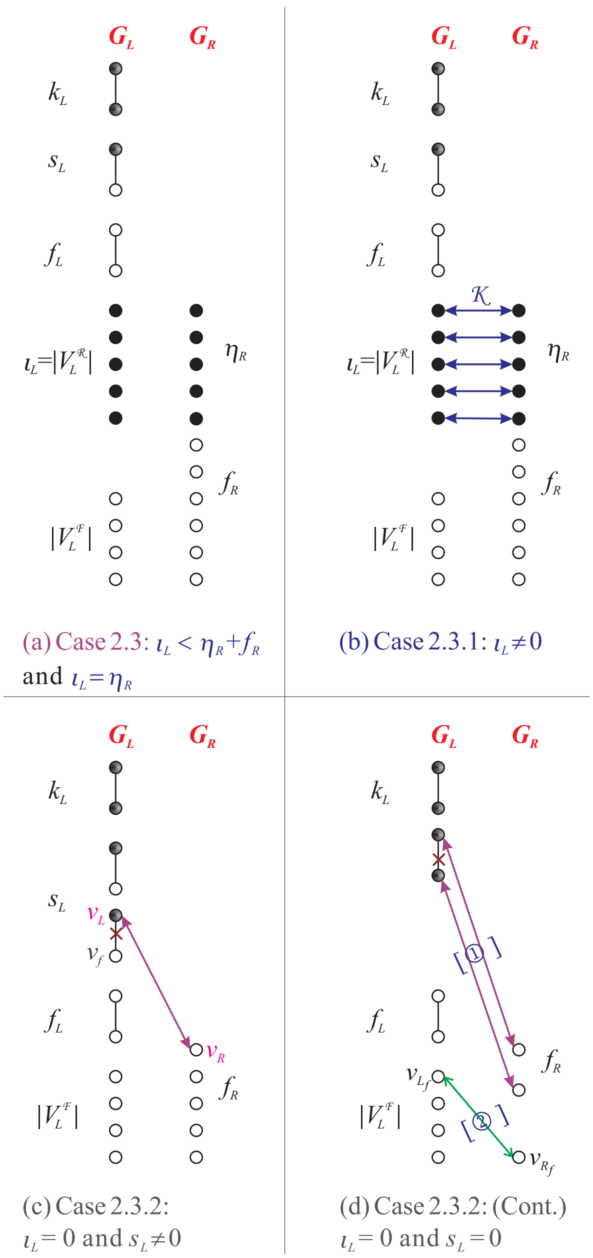
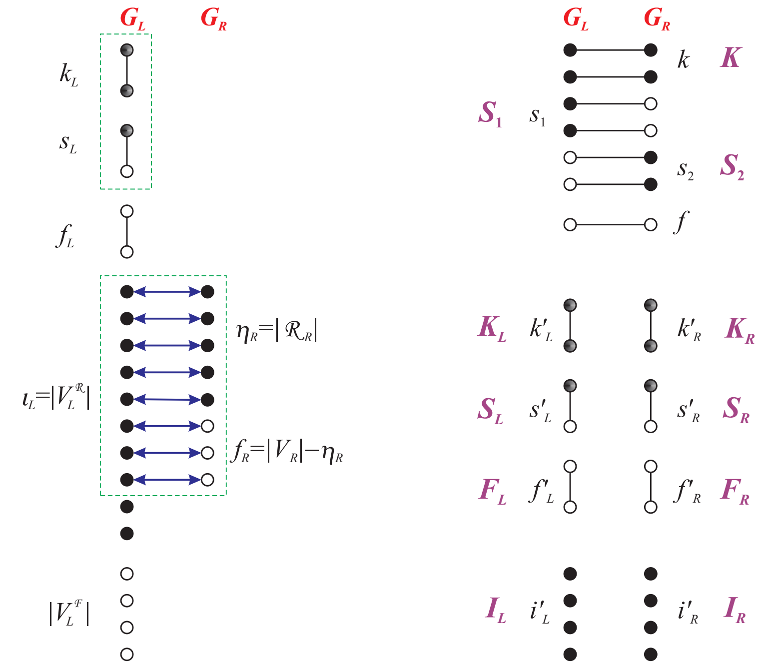


Fig. 6: The construction of a matched-paired-dominating set $CMPD$ of G for Case 2.3, where (a) the partition of V_L and V_R for the case, (b) the construction of a matched-paired-dominating set for case of $\imath_L \neq 0$, and (c)–(d) the construction of a matched-paired-dominating set for case of $\imath_L = 0$. Note that restricted vertices are drawn by filled circles, symbol '×' denotes the destruction to one paired-edge in $CMPD_L$, and arrow lines represent the new paired-edges for the construction.



(a) (b)

Fig. 7: (a) The construction of a matched-paired-dominating set $CMPD$ of G for case of $\imath_L \geqslant \eta_R + f_R$, and (b) the possible paired-edges in a canonical matched-paired-dominating set $MMPD$ of G with the largest number of mixed paired-edges, where restricted vertices are drawn by filled circles and arrow lines represent the new paired-edges for the construction.

$V'_L = V_L - I(G_L) - V(MMPD_L)$. For $v'_L \in V'_L$, if v'_L is not dominated by $V(MMPD_L)$, then let $v''_L \in N_{G_L}(v'_L) - V(MMPD_L)$, $MMPD_L = MMPD_L \cup \{\langle v'_L, v''_L\rangle\}$, and let $V'_L = V'_L - \{v'_L, v''_L\}$; otherwise, let $V'_L = V'_L - \{v'_L\}$. Since v'_L is not an isolated vertex in G_L, v''_L does exist if v'_L is not dominated by $V(MMPD_L)$. Repeat the above process until $V'_L = \emptyset$. Then, $MMPD_L$ is a matched-paired-dominating set of $G_L - I(G_L)$ satisfying that $|V(MMPD_L) \cap (\mathcal{R}_L - I(G_L))| \geqslant 2k'_L + s'_L$. Since $CMPD_L$ is a maximum (k_L, s_L, f_L)-matched-paired-dominating set of $G_L - I(G_L)$, $|V(CMPD_L) \cap (\mathcal{R}_L - I(G_L))| = 2k_L + s_L \geqslant |V(MMPD_L) \cap (\mathcal{R}_L - I(G_L))| \geqslant 2k'_L + s'_L$.

Next, we prove Claim (2) that $i'_R = s_2 = k'_R = s'_R = 0$. We first show that $i'_R = 0$. Assume by contradiction that $i'_R \neq 0$. Then, $i'_L = 0$. By Statements (1) and (4) of Lemma 5, $s_1 = f = s'_L = f'_L = 0$. By assumption, $|\mathcal{R}_L| = k + 2k'_L > |\mathcal{R}_R| = k + s_2 + 2k'_R + s'_R + i'_R$. Thus, $2k'_L > s_2 + 2k'_R + s'_R + i'_R \geqslant 1$ and, hence, $k'_L \geqslant 1$. Let v_R be a restricted vertex in I_R and let $\langle v_L, v'_L\rangle$ be a full-paired-edge in K_L. By pairing v_L with v_R and all the other paired-edges stay the same, we obtain a maximum matched-paired-dominating set $MMPD'$ of G having more mixed paired-edges than $MMPD$, a contradiction. Thus, $i'_R = 0$. We then prove that $s_2 = k'_R = s'_R = 0$. Assume by contradiction that $s_2 + k'_R + s'_R \neq 0$. By assumption, $|\mathcal{R}_L| > |\mathcal{R}_R|$.

Then, $k + s_1 + 2k'_L + s'_L + i'_L > k + s_2 + 2k'_R + s'_R$. Thus, $2k'_L + s_1 + s'_L + i'_L > 2k'_R + s_2 + s'_R$. Let R be the set of restricted vertices in $S_2 \cup K_R \cup S_R$. Then, $|R| = 2k'_R + s_2 + s'_R$. Suppose that $2k'_L < 2k'_R + s_2 + s'_R$. Let L be a subset of restricted vertices in $S_1 \cup S_L \cup I_L$ such that $|L| = (2k'_R + s_2 + s'_R) - 2k'_L$. Then, $|L| + |V(K_L)| = 2k'_R + s_2 + s'_R = |R|$. By pairing every vertex in R with one restricted vertex of $V(K_L) \cup L$ and all the other paired-edges stay the same, we obtain a maximum matched-paired-dominating set $MMPD'$ of G having more mixed paired-edges than $MMPD$, a contradiction. In the following, suppose that $2k'_L \geqslant 2k'_R + s_2 + s'_R$. Consider the following cases:

Case 1: $2k'_R + s_2 + s'_R$ is even. Let $K_L = K_L^a \cup K_L^b$ such that $K_L^a \cap K_L^b = \emptyset$ and $|K_L^a| = \frac{2k'_R + s_2 + s'_R}{2}$. By pairing every vertex in R with one restricted vertex of $V(K_L^a)$ and all the other paired-edges stay the same, we obtain a maximum matched-paired-dominating set $MMPD'$ of G having more mixed paired-edges than $MMPD$, a contradiction.

Case 2: $2k'_R + s_2 + s'_R$ is odd. Let $K_L = K_L^a \cup K_L^b$ such that $K_L^a \cap K_L^b = \emptyset$ and $|K_L^a| = \lfloor \frac{2k'_R + s_2 + s'_R}{2} \rfloor$. Consider the following subcases:

Case 2.1: $s_1 + s'_L + i'_L \neq 0$. Let v_L be a restricted vertex in $S_1 \cup S_L \cup I_L$. Let v_R be a vertex in R and let $R' = R - \{v_R\}$. Then, $\frac{|R'|}{2} = \lfloor \frac{2k'_R + s_2 + s'_R}{2} \rfloor$. By pairing v_R with v_L, pairing every vertex in R' with one restricted vertex of $V(K_L^a)$, and all the other paired-edges stay the same, we obtain a maximum matched-paired-dominating set $MMPD'$ of G having more mixed paired-edges than $MMPD$, a contradiction.

Case 2.2: $s_1 + s'_L + i'_L = 0$. Suppose that $s'_R \neq 0$. Let $\langle v_R, v_f \rangle$ be a semi-paired-edge, with restricted vertex v_R, in S_R. Let $R' = R - \{v_R\}$ and let $\langle v_L, v'_L \rangle$ be a full-paired-edge in K_L^b. By pairing v_R with v_L, pairing v_f with v'_L, pairing every vertex in R' with one restricted vertex of $V(K_L^a)$, and all the other paired-edges stay the same, we obtain a maximum matched-paired-dominating set $MMPD'$ of G having more mixed paired-edges than $MMPD$, a contradiction. On the other hand, suppose that $s'_R = 0$. Then, s_2 is odd. We prove $f_R \neq 0$. Assume by contradiction that $f_R = 0$. By assumption of the lemma, $\imath_L \geqslant \eta_R = |V_R| = k + s_2 + 2k'_R$. Since $s_2 > 0$, $\imath_L \geqslant k + 1$ and, hence, $\imath_L - k > 0$. Then, $|\mathcal{R}_L| = 2k_L + s_L + \imath_L = k + 2k'_L$. Consequently, $(2k_L + s_L) - 2k'_L = k - \imath_L < 0$. It contradicts that $(2k_L + s_L) - 2k'_L \geqslant 0$ by Claim (1). Thus, $f_R \neq 0$. Let v_f be a free vertex in V_R, $\langle v_R, v'_f \rangle$ be a semi-paired-edge in S_2 such that $v_R \in \mathcal{R}_R$, $R' = R - \{v_R\}$, and let $\langle v_L, v'_L \rangle$ be a full-paired-edge in K_L^b. By pairing v_R with v_L, pairing v_f with v'_L, pairing every vertex in R' with one restricted vertex of $V(K_L^a)$, and all the other paired-edges stay the same, we obtain a maximum matched-paired-dominating set $MMPD'$ of G having more mixed paired-edges than $MMPD$, a contradiction.

It follows from the above arguments that Claim (2) holds true; i.e., $i'_R = s_2 = k'_R = s'_R = 0$.

Thus, $k = \eta_R$. Suppose that $i'_L \neq 0$. Then, $f = f'_R = 0$ by Statement (1) of Lemma 5. Assume by contradiction that $\widehat{f}_R = f_R - s_1 \neq 0$. Then, $|\mathcal{R}_L| = k + s_1 + 2k'_L + s'_L + i'_L = 2k_L + s_L + \imath_L \geqslant 2k_L + s_L + \eta_R + s_1 + \widehat{f}_R = 2k_L + s_L + k + s_1 + \widehat{f}_R$. By Claim (1), $2k_L + s_L \geqslant 2k'_L + s'_L$. Thus, $i'_L \geqslant \widehat{f}_R$. By pairing every free vertex in $V_R - V(MMPD)$ with one restricted vertex in I_L and all the other paired-edges stay the same, we obtain a matched-paired-dominating set of G having more restricted vertices than $MMPD$, a contradiction. Thus, $f_R = s_1$. On the other hand, suppose that $i'_L = 0$. By Claim (1), $2k_L + s_L \geqslant 2k'_L + s'_L$. By assumption of the lemma, $\imath_L \geqslant |V_R| = k + s_1 + f_R - s_1$. Then, $|\mathcal{R}_L| = k + s_1 + 2k'_L + s'_L = 2k_L + s_L + \imath_L \geqslant 2k_L + s_L + k + s_1 + f_R - s_1$. Thus, $2k'_L + s'_L \geqslant 2k_L + s_L + f_R - s_1$. Since $2k_L + s_L \geqslant 2k'_L + s'_L$ by Claim (1), $f_R - s_1 = 0$. Thus, $f_R = s_1$. Consequently, $k = \eta_R$ and $s_1 = f_R$. We can see that $|V(CMPD) \cap \mathcal{R}| = 2k_L + s_L + 2\eta_R + f_R = 2k_L + s_L + 2k + s_1$ and $|V(MMPD) \cap \mathcal{R}| = 2k'_L + s'_L + 2k + s_1$. Thus, $|V(CMPD) \cap \mathcal{R}| - |V(MMPD) \cap \mathcal{R}| = (2k_L + s_L) - (2k'_L + s'_L) \geqslant 0$ by Claim (1). That is, the constructed matched-paired-dominating set $CMPD$ is a maximum matched-paired-dominating set of G. In addition, $CMPD$ contains no free-paired-edge. Thus, the constructed $(k_L + \eta_R, s_L + f_R, 0)$-matched-paired-dominating set $CMPD$ is a canonical matched-paired-dominating set of G. $\square$

It follows from Lemma 8 that our constructed matched-paired-dominating set $CMPD$ is a canonical matched-paired-dominating set of G w.r.t. $\mathcal{R}$. Now, we will analyze the time complexity for constructing $CMPD$. For case of $\imath_L \geqslant \eta_R + f_R$ shown in Fig. 7(a), $CMPD$ is constructed in $O(|V_R|)$ time, where $|V_R| \leqslant |\mathcal{R}_L|$. Consider that $\imath_L < \eta_R + f_R$. For case of $\eta_R < \imath_L$ shown in Fig. 5(a), $CMPD$ is constructed in $O(\imath_L)$ time, where $\imath_L \leqslant |\mathcal{R}_L|$. For case of $\eta_R > \imath_L$ shown in Fig. 5(b)–(d), $CMPD$ can be easily constructed in $O(|\mathcal{R}_R|)$ time, where $|\mathcal{R}_R| \leqslant |\mathcal{R}_L|$. On the other hand, for case of $\imath_L = \eta_R$ shown in Fig. 6, $CMPD$ can be constructed in $O(|\mathcal{R}_R|)$ time, where $|\mathcal{R}_R| \leqslant |\mathcal{R}_L|$. It follows from the above arguments that constructing a canonical matched-paired-dominating set $CMPD$ of G runs in $O(|\mathcal{R}_L|)$ time. Let $\widehat{E}_{LR} = \{uv | \forall u \in V_L$ and $\forall v \in V_R\}$. Then, $|\mathcal{R}_L| \leqslant |\widehat{E}_{LR}|$. Hence, a canonical matched-paired-dominating set $CMPD$ of G can be computed in $O(|\widehat{E}_{LR}|)$ time.

It follows from the above analysis that given a decomposition tree of a cograph $G = (V, E)$ and a restricted vertex set $\mathcal{R} \subseteq V$, a canonical matched-paired-dominating set of G w.r.t. $\mathcal{R}$ can be constructed in $O(|V| + |E|)$-linear time. Thus, we conclude the following theorem.

Theorem 9. *Given a cograph $G = (V, E)$ with restricted vertex set $\mathcal{R}$, the maximum matched-paired-domination problem can be solved in $O(|V| + |E|)$-linear time.*

4 Concluding Remarks

The paired-domination problem can be applied to allocate guards on vertices such that these guards protect every vertex, each guard is assigned another adjacent one, and they are designed as backup for each other. However, some vertices may play more important role (for example, important facilities are placed on these vertices) and, hence, they are placed by guards for instant protection possible. Motivated by the issue we propose a generalization of the paired-domination problem, namely, the maximum matched-paired-domination problem. We then solve the maximum matched-paired-domination problem on cographs in linear time. A future work will be to extend our technique to solve the maximum matched-paired-domination problem on some special classes of graphs, such as trees, block graphs, Ptolemaic graphs and distance-hereditary graphs.

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List of Symbols

1. $N_G(v)$, $N_G[v]$: $N_G(v)$ is the *open neighborhood* of a vertex v in a graph $G = (V, E)$ and is defined to be $\{u \in V | uv \in E\}$. $N_G[v]$ is the *closed neighborhood* of v and is defined to be $N_G(v) \cup \{v\}$.

2. $G[S]$: the subgraph of G induced by the vertices in S, where S is a subset of vertices of G.

3. *matching*, *perfect matching*: A matching in a graph G is a set of independent edges in G. A perfect matching M in a graph G is a matching such that every vertex of G is incident to an edge of M.

4. *paired-dominating set*: A set PD of vertices of G is a paired-dominating set of G if PD is a dominating set of G and if $G[PD]$ contains at least one perfect matching.

5. *paired-domination number* $\gamma_{\mathrm{p}}(G)$: is the minimum cardinality of a paired-dominating set for a graph G.

6. *minimum paired-dominating set*: is a paired-dominating set of G with cardinality $\gamma_{\mathrm{p}}(G)$.

7. $V(M)$: For a set M of independent edges in a graph, $V(M)$ denotes the set of vertices being incident to edges of M.

8. *matched-paired-dominating set*: A set MPD of independent edges in a graph G is a matched-paired-dominating set of G if MPD is a perfect matching of $G[PD]$ induced by a paired-dominating set PD of G. Note that $V(MPD)$ is a paired-dominating set PD of G and MPD specifies a perfect matching of $G[PD]$.

9. *restricted vertex set* $\mathcal{R}$: The restricted vertex set $\mathcal{R}$ is a subset of vertices in a graph and is a part of the input for the proposed problem in the paper. Any vertex in $\mathcal{R}$ is called *restricted vertex* and the other is called *free vertex*.

10. *maximum matched number* $\beta(G)$: For a matched-paired-dominating set MPD of G, the matched number of MPD is defined to be $|V(MPD) \cap \mathcal{R}|$. The maximum matched number $\beta(G)$ of G is the largest matched number of a matched-paired-dominating set of G.

11. *maximum matched-paired-dominating set*: is a matched-paired-dominating set of a graph G with matched number $\beta(G)$.

12. *paired-edge* $\langle u, v\rangle$: is an element in a matched-paired-dominating set MPD of a graph. We call u the partner of v in MPD. A paired-edge in MPD is called *full-paired-edge* if both of its vertices are in $\mathcal{R}$, is called *semi-paired-edge* if its one vertex is in $\mathcal{R}$ but the other vertex is not in $\mathcal{R}$, and is called *free-paired-edge* if both of its vertices are not in $\mathcal{R}$.

13. *canonical matched-paired-dominating set*: is a maximum matched-paired-dominating set of a graph G with the least free-paired-edges.

14. *maximum matched-paired-domination problem*: Given a graph G and a subset $\mathcal{R}$ of vertices in G, the problem is to find a canonical matched-paired-dominating set of G. Note that the proposed problem is a generalization of the paired-domination problem and it coincides with the classical paired-domination problem if $\mathcal{R} = \emptyset$.

15. (k, s, f)-matched-paired-dominating set: is a matched-paired-dominating set MPD of G w.r.t. $\mathcal{R}$ satisfying that (1) $|MPD| = k + s + f$; (2) there are exactly k full-paired-edges in MPD; (3) there are exactly s semi-paired-edges in MPD, and (4) all other paired-edges in MPD are free-paired-edges.

16. $K_G(MPD)$, $S_G(MPD)$, $F_G(MPD)$: For a (k, s, f)-matched-paired-dominating set MPD of a graph G, $K_G(MPD)$, $S_G(MPD)$, and $F_G(MPD)$ are defined to be the subsets of MPD consisting of all full-paired-edges, all semi-paired-edges, and all free-paired-edges in MPD, respectively. That is, $|K_G(MPD)| = k$, $|S_G(MPD)| = s$, and $|F_G(MPD)| = f$.

17. $G = G_L \oplus G_R$: $G = (V_L \cup V_R, E_L \cup E_R)$ is formed from $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ by a *union operation*.

18. $G = G_L \otimes G_R$: $G = (V, E)$ is formed from $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ by a *joint operation*, where $V = V_L \cup V_R$ and $E = E_L \cup E_R \cup \{uv | \forall u \in V_L \text{ and } \forall v \in V_R\}$.

19. $I(G)$: is the set of isolated vertices in graph G.

20. *mixed paired-edges*: Let $G = G_L \otimes G_R$. A paired-edge in a matched-paired-dominating set of G is called *mixed* if one of its vertices is in G_L and the other is in G_R.

21. $\imath_L, \eta_R, f_R$: Let $G = G_L \otimes G_R$ with restricted vertex set $\mathcal{R}$, $\mathcal{R}_L = \mathcal{R} \cap V_L$, $\mathcal{R}_R = \mathcal{R} \cap V_R$, and let $CMPD_L$ be a canonical (k_L, s_L, f_L)-matched-paired-dominating set of $G_L - I(G_L)$. Define $\imath_L = |\mathcal{R}_L - V(CMPD_L)|$, $\eta_R = |\mathcal{R}_R|$, and $f_R = |V_R| - \eta_R$.